Int. J. Solids Structures Vol. 22, No. 7, pp. 759–766, 1986
Printed in Great Britain.

0020-7683/86 \$3.00+.00
Pergamon Journals Ltd.

A CONTOUR INTEGRAL AND AN ENERGY RELEASE RATE IN THERMOELASTICITY

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(*Received* 11 *February* 1985)

Abstract—This paper is a sequel to Ref. [1] in which a material momentum for thermoelasticity was derived. Our main goal here is to show that, in analogy to the elastic case, it is possible to relate an integral of the material momentum with a concept of energy release rate. However the relation obtained is not local in time: it holds during a time interval, which agrees with the results obtained in Ref. [1].

1. INTRODUCTION

In a previous paper[1] we constructed a Lagrangian density L_{t_0} for linear thermoelasticity (eqn (8) of Ref. [1]).†

From its expression we generated quantities that were found to be divergence-free in any part of a thermoelastic body without defects. This divergence was to be understood in *both* space and time. These quantities, called *material momenta* were integrated over a time period extending from time 0 to a specified final time t_0. We then proceeded to integrate the quantities obtained on a subdomain G and we produced a material analogue of an impulse-momentum type relation (eqn (22) of Ref. [1])

$$\int_{\partial G} I_{mj}(\mathbf{a}, t_0) n_j \, \mathrm{d}\sigma = -\left\{\int_G \rho \dot{x}_i(\mathbf{a}, t) x_{i,m}(\mathbf{a}, t_0 - t) \, \mathrm{d}^3 a\right\}_{t=0}^{t=t_0} - \int_G s(\mathbf{a}, 0)\eta_{,m}(\mathbf{a}, t_0) \, \mathrm{d}^3 a, \quad (1)$$

where, as in Ref. [1], $\mathbf{a}$ is the material coordinate, $x(\mathbf{a}, t)$ the position of the particle originating from $\mathbf{a}$, $s(\mathbf{a}, t)$ the entropy, and where, if $\tau(\mathbf{a}, t)$ denotes the temperature increment,

$$\eta(\mathbf{a}, t) = \int_0^t \tau(\mathbf{a}, t') \, \mathrm{d}t'. \quad (2)$$

Finally, $I_{mj}(\mathbf{a}, t)$ is given in eqn (20) of Ref. [1].

In this paper we show that, in analogy to the elastic case, the quantity

$$J_m^{t_0} = \int_C I_{mj}(\mathbf{a}, t_0) n_j \, \mathrm{d}\sigma, \quad (3)$$

where C is any smooth surface around a defect in a given body B, is related to the concept of energy release rate (over the time interval $(0, t_0)$) of the body B, due to an infinitesimal translation of the defect in the mth-direction.

† In eqn (8) of Ref. [1], the term $\frac{1}{4}\lambda_{ij}(a)[\dot{\eta}_{,j}, \dot{\eta}_{,i}]_{t_0}^{+}$ should read as $\frac{1}{4}\lambda_{ij}(a)[\dot{\eta}_{,j}, \eta_{,i}]_{t_0}^{+}$, while in eqn (20) of Ref. [1], the term $L_{t_0}\delta_{jm}$ should read $-L_{t_0}\delta_{jm}$.

In quasistatic elasticity, the quantity

$$J_m = \int_C -\{L_{jm} - \sigma_{ij}(\mathbf{a}, t)x_{i,m}(\mathbf{a}, t)\}n_j \, d\sigma, \tag{4}$$

where L is the classical Lagrangian functional for elasticity, is well known to be equal to the potential energy release rate associated with the quasi-static virtual translation of C in the mth-direction[2].

Since critical values of the energy release rate determine the growth or motion of the defect, J_m is indeed an essential quantity. Reference is made to Herrmann[3] or Budiansky and Rice[4] for further details.

We propose to show that, in the case of linearized thermoelasticity, the quantity $J_m^{t_0}$ defined by (3) equals the potential energy release rate over the time interval $(0, t_0)$ of the body B, due to a *purely elastic quasi-static virtual* transformation of this body. The details of this transformation will be described later on in this paper.

To this effect we first derive a complete expression for the variation δA_G of the action integral

$$A_G = \int_0^{t_0} \int_G L_{t_0} \, d^3a \, dt, \tag{5}$$

under a one-parameter family of transformations for both dependent and independent variables (Section 2). In (5), G is the subdomain of B lying between C and ∂B, the *external* boundary of B.

We then proceed to derive the potential energy release rate associated with a purely elastic quasistatic virtual transformation of the body under consideration (Section 3).

Upon postulating stationarity of the action under the one-parameter family of transformations considered in Section 2, we obtain in Section 4 a so-called "transversality relation" between the variation of the dependent and independent variables. A simple inspection of this relation enables us to make our conclusions in Sections 4 and 5, based on developments presented in Section 3.

The reader is referred to [1] for the definition of all undefined quantities that appear in the text.

2. VARIATION OF THE ACTION FUNCTIONAL

It is assumed here that the body under consideration is homogeneous. Furthermore, all thermal sources or body loadings are neglected. We then consider a one parameter family of transformations $\tilde{a}_k(\lambda, t, a_m)$, $\tilde{x}_k(\lambda, t, a_m)$, $\tilde{\eta}(\lambda, t, a_m)$ of both dependent and independent variables with the following properties:

—at $\lambda = 0$, virtual and real variables coincide, i.e.

$$\tilde{a}_k(0, t, a_m) = a_k, \quad \tilde{x}_k(0, t, a_m) = x_k(t, a_m), \quad \tilde{\eta}(0, t, a_m) = \eta(t, a_m), \tag{6}$$

—the virtual dependent fields $(\tilde{x}_k, \tilde{\eta})$ are authentic fields, i.e.

$$\tilde{x}_k(\lambda, t, a_m) = \tilde{x}_k(t, \tilde{a}_m), \quad \tilde{\eta}(\lambda, t, a_m) = \tilde{\eta}(t, \tilde{a}_m). \tag{7}$$

Hypotheses (6) and (7) imply that all derivatives of the virtual fields $\tilde{x}_k$, $\tilde{\eta}$ with respect to the virtual independent variables $\tilde{a}_m$, t coincide with their real analogues for $\lambda = 0$, i.e. for example,

$$\left.\frac{\partial \tilde{\eta}}{\partial \tilde{a}_j}\right|_{\lambda=0} = \frac{\partial \eta}{\partial a_j}, \ldots \tag{8}$$

We set

$$\left.\frac{\mathrm{d}\tilde{a}_k}{\mathrm{d}\lambda}\right|_{\lambda=0} = \delta a_k, \quad \left.\frac{\mathrm{d}\tilde{x}_k}{\mathrm{d}\lambda}\right|_{\lambda=0} = \delta x_k, \quad \left.\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}\lambda}\right|_{\lambda=0} = \delta\eta, \tag{9}$$

and abbreviate $\delta a_k(a_l, t)$ to $\delta a_k(t), \ldots$.

Then, for example,

$$\tilde{x}_k = x_k + \lambda\,\delta x_k + \mathrm{o}(\lambda). \tag{10}$$

We finally assume that, on the part C of the boundary of G, $\delta a_k(t)$ is specified, and that on δB, all δa_k's are equal to 0.

Call $\tilde{G}$ the image of G under this transformation. The Jacobian of the transformation is given by

$$\mathrm{d}^3\tilde{a}\,\mathrm{d}t = \left(1 + \lambda\frac{\partial}{\partial a_k}\delta a_k + \mathrm{o}(\lambda)\right)\mathrm{d}^3 a\,\mathrm{d}t. \tag{11}$$

The variation of A_G, denoted δA_G, is such that

$$\begin{aligned}-\delta a_G = \lim_{\lambda\to 0}\frac{A_{\tilde{G}} - A_G}{\lambda} = \int_0^{t_0}\int_G \Big\{ & -L_{t_0}\frac{\partial}{\partial a_k}\delta a_k(t) \\ & + \sigma_{ij}(\mathbf{a}, t)\delta x_{i,j}(t_0 - t) - s(\mathbf{a}, t)\delta\dot{\eta}(t_0 - t) \\ & + \rho\dot{x}_i(\mathbf{a}, t)\delta\dot{x}_i(t_0 - t) - \tfrac{1}{2}q_i(\mathbf{a}, t)\delta\eta_{,i}(t_0 - t) \\ & - \tfrac{1}{2}\lambda_{ij}\eta_{,j}(\mathbf{a}, t)\delta\dot{\eta}_{,i}(t_0 - t)\Big\}\,\mathrm{d}^3 a.\end{aligned} \tag{12}$$

In the derivation of (12) we have taken advantage of the assumptions of homogeneity as well as of the symmetry of the convolution operator. The $\delta x_{i,j}$'s, $\delta\dot{x}_i$'s, $\delta\dot{\eta}$'s, $\delta\eta_{,j}$'s and $\delta\dot{\eta}_{,j}$'s are to be understood as

$$\delta x_{i,j}(t) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{\partial\tilde{x}_i}{\partial\tilde{a}_j}\right]_{\lambda=0}(t), \ldots. \tag{13}$$

We now proceed to express these quantities in terms of δa_k, $\delta\eta$, δx_i. Since these computations are somewhat tedious, not all derivations are presented. Only the most intricate, that of $\delta\dot{\eta}_{,j}$, is given. The principle would be the same for all others. Whenever we use ∂/∂ here, we mean differentiation with respect to the explicit dependence; otherwise we use d/d. Consider

$$\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}\lambda} = \frac{\partial\tilde{\eta}}{\partial\tilde{a}_k}\frac{\mathrm{d}\tilde{a}_k}{\mathrm{d}\lambda},$$

then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}\lambda}\right) = \frac{\partial^2\tilde{\eta}}{\partial\tilde{a}_k\,\partial\tilde{a}_m}\frac{\partial\tilde{a}_m}{\partial t}\frac{\mathrm{d}\tilde{a}_k}{\mathrm{d}\lambda} + \frac{\partial^2\tilde{\eta}}{\partial\tilde{a}_k\,\partial t}\frac{\mathrm{d}\tilde{a}_k}{\mathrm{d}\lambda} + \frac{\partial\tilde{\eta}}{\partial\tilde{a}_k}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\tilde{a}_k}{\mathrm{d}\lambda}\right), \tag{14}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t\,\mathrm{d}a_j}\left(\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}\lambda}\right) = \frac{\partial^3\tilde{\eta}}{\partial\tilde{a}_k\,\partial\tilde{a}_m\,\partial\tilde{a}_l}\frac{\partial\tilde{a}_l}{\partial a_j}\frac{\partial\tilde{a}_m}{\partial t}\frac{\mathrm{d}\tilde{a}_k}{\mathrm{d}\lambda} + \frac{\partial^2\tilde{\eta}}{\partial\tilde{a}_k\,\partial\tilde{a}_m}\frac{\partial^2\tilde{a}_m}{\partial t\,\partial a_j}\frac{\mathrm{d}\tilde{a}_k}{\mathrm{d}\lambda} + \frac{\partial^2\tilde{\eta}}{\partial\tilde{a}_k\,\partial\tilde{a}_m}\frac{\partial\tilde{a}_m}{\partial t}\frac{\mathrm{d}}{\mathrm{d}a_j}\left(\frac{\mathrm{d}\tilde{a}_k}{\mathrm{d}\lambda}\right)$$

$$+\frac{\partial^3\tilde{\eta}}{\partial\tilde{a}_k\,\partial\tilde{a}_l\,\partial t}\frac{\partial\tilde{a}_l}{\partial a_j}\frac{\mathrm{d}\tilde{a}_k}{\mathrm{d}\lambda} + \frac{\partial^2\tilde{\eta}}{\partial\tilde{a}_k\,\partial t}\frac{\mathrm{d}}{\mathrm{d}a_j}\left(\frac{\mathrm{d}\tilde{a}_k}{\mathrm{d}\lambda}\right) + \frac{\partial^2\tilde{\eta}}{\partial\tilde{a}_k\,\partial\tilde{a}_l}\frac{\partial\tilde{a}_l}{\partial a_j}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\tilde{a}_k}{\mathrm{d}\lambda}\right) + \frac{\partial\tilde{\eta}}{\partial\tilde{a}_k}\frac{\mathrm{d}^2}{\mathrm{d}t\,\mathrm{d}a_j}\left(\frac{\mathrm{d}\tilde{a}_k}{\mathrm{d}\lambda}\right). \quad (15)$$

At $\lambda = 0$, we obtain, in view of (8),

$$\frac{\mathrm{d}^2}{\mathrm{d}t\,\mathrm{d}a_j}(\delta\eta) = \frac{\partial^3\eta}{\partial a_k\,\partial a_l\,\partial t}\delta_{lj}\,\delta a_k + \frac{\partial^2\eta}{\partial a_k\,\partial t}\frac{\mathrm{d}}{\mathrm{d}a_j}(\delta a_k)$$
$$+\frac{\partial^2\eta}{\partial a_k\,\partial a_l}\delta_{lj}\frac{\mathrm{d}}{\mathrm{d}t}(\delta a_k) + \frac{\partial\eta}{\partial a_k}\frac{\mathrm{d}^2}{\mathrm{d}t\,\mathrm{d}a_j}(\delta a_k). \quad (16)$$

But

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial^2\tilde{\eta}}{\partial t\,\partial\tilde{a}_j}\right) = \frac{\partial^3\tilde{\eta}}{\partial t\,\partial\tilde{a}_j\,\partial\tilde{a}_k}\frac{\mathrm{d}\tilde{a}_k}{\mathrm{d}\lambda}, \quad (17)$$

thus at $\lambda = 0$,

$$\delta\dot{\eta}_{,j} = \frac{\partial^3\eta}{\partial t\,\partial a_j\,\partial a_k}\delta a_k, \quad (18)$$

which is precisely the first term of the right-hand side of (16). Relation (16) can be rewritten as

$$\delta\dot{\eta}_{,j} = \frac{\partial^2}{\partial t\,\partial a_j}(\delta\eta) - \dot{\eta}_{,k}\frac{\partial}{\partial a_j}(\delta a_k) - \eta_{,k}\frac{\partial^2}{\partial t\,\partial a_j}(\delta a_k) - \eta_{,kj}\frac{\partial}{\partial t}(\delta a_k). \quad (19)$$

Since we do not further refer to explicit differentiation, we abandon the distinction previously introduced. Similarly, we would obtain the following expressions for $\delta x_{i,j}$, $\delta\eta_{,j}$, $\delta\dot{x}_i$, $\delta\dot{\eta}$:

$$\begin{aligned}
\delta x_{i,j} &= \frac{\partial}{\partial a_j}\delta x_i - x_{i,k}\frac{\partial}{\partial a_j}\delta a_k \\
\delta\eta_{,j} &= \frac{\partial}{\partial a_j}\delta\eta - \eta_{,k}\frac{\partial}{\partial a_j}\delta a_k \\
\delta\dot{x}_i &= \frac{\partial}{\partial t}\delta x_i - x_{i,k}\frac{\partial}{\partial t}\delta a_k \\
\delta\dot{\eta} &= \frac{\partial}{\partial t}\delta\eta - \eta_{,k}\frac{\partial}{\partial t}\delta a_k.
\end{aligned} \quad (20)$$

We now use (19) and (20) in (12) and integrate by parts appropriately. With the notations of Ref. [1], we obtain

$$\left.\begin{aligned}
-\delta A_G = \int_G \mathrm{d}^3a \int_0^{t_0} &\mathrm{d}t[-\sigma_{ij,j} + \rho\ddot{x}_i](\mathbf{a}, t)\,\delta x_i(t_0 - t) \\
&+(-\dot{s} + q_{i,i})(\mathbf{a}, t)\,\delta\eta(t_0 - t) - \{\partial_j[-L_{t_0}\delta_{jl} - \sigma_{ij}(\mathbf{a}, t)x_{i,l}(\mathbf{a}, t_0 - t) \\
&+q_j(\mathbf{a}, t)\eta_{,l}(\mathbf{a}, t_0 - t)] + \partial_t[\rho\dot{x}_i(\mathbf{a}, t)x_{i,l}(\mathbf{a}, t_0 - t) \\
&-s(\mathbf{a}, t)\eta_{,l}(\mathbf{a}, t_0 - t) - \tfrac{1}{2}\lambda_{ij}\eta_{,j}(\mathbf{a}, t)\eta_{,il}(\mathbf{a}, t_0 - t)]\}\,\delta a_l(t_0 - t)
\end{aligned}\right\} \quad (21a)$$

$$
\left.\begin{aligned}
&+\partial_j\{[-L_{t_0}\delta_{jl}-\sigma_{ij}(\mathbf{a},t)x_{i,l}(\mathbf{a},t_0-t)\\
&+q_j(\mathbf{a},t)\eta_{,l}(\mathbf{a},t_0-t)]\delta a_l(t_0-t)+\sigma_{ij}(\mathbf{a},t)\delta x_i(t_0-t)\\
&-q_j(\mathbf{a},t)\delta\eta(t_0-t)\}\\
&+\partial_t\{[\rho\dot{x}_i(\mathbf{a},t)x_{i,l}(\mathbf{a},t_0-t)-s(\mathbf{a},t)\eta_{,l}(\mathbf{a},t_0-t)\\
&-\tfrac{1}{2}\lambda_{ij}\eta_{,j}(\mathbf{a},t)\eta_{,il}(\mathbf{a},t_0-t)]\delta a_l(t_0-t)-\rho\dot{x}_i(\mathbf{a},t)\delta x_i(t_0-t)\\
&+s(\mathbf{a},t)\delta\eta(t_0-t)+\tfrac{1}{2}\lambda_{ij}\eta_{,j}(\mathbf{a},t)\partial_i[\delta\eta(t_0-t)]\}\\
&-\partial_l\{\tfrac{1}{2}\lambda_{ij}\eta_{,j}(\mathbf{a},t)\eta_{,l}(\mathbf{a},t_0-t)\partial_i[\delta a_l(t_0-t)]\}
\end{aligned}\right\} \tag{21b}
$$

$$
+L_{t_0}\delta_{jl}\,\partial_j[\delta a_l(t_0-t)-\delta a_l(t)]. \tag{21c}
$$

It is appropriate at this point to further specify the transformation: δa_l is *independent of t* and it reduces to a translation in the mth-direction on C, i.e.

$$
\delta a_l=\delta_{lm}\ \text{on}\ C. \tag{22}
$$

Under such a transformation, (21) simplifies considerably. The bracket (a) vanishes. Indeed, its first two terms are the equations of motion and its third term is precisely the material conservation law obtained in Ref. [1] (eqn (19) of Ref. [1]). The bracket (c) also vanishes, since δa_l does not depend on t. We apply the divergence theorem on (b), keeping in mind that on ∂B all material virtual transformations vanish. We obtain, in view of (3),

$$
\left.\begin{aligned}
J_m^{t_0}+\int_G\Big|&\{[\rho\dot{x}_i(\mathbf{a},t)x_{i,l}(\mathbf{a},t_0-t)-s(\mathbf{a},t)\eta_{,l}(\mathbf{a},t_0-t)\\
&-\tfrac{1}{2}\lambda_{ij}\eta_{,j}(\mathbf{a},t)\eta_{,il}(\mathbf{a},t_0-t)]\,\delta a_l\\
&-\tfrac{1}{2}\lambda_{ij}\eta_{,j}(\mathbf{a},t)\eta_{,l}(\mathbf{a},t_0-t)\,\partial_i\delta a_l\}\Big|_{t=0}^{t=t_0}\mathrm{d}^3a
\end{aligned}\right\} \tag{23a}
$$

$$
\left.\begin{aligned}
&+\int_{\partial G}\int_0^{t_0}[\sigma_{ij}(\mathbf{a},t)\delta x_i(t_0-t)-q_j(\mathbf{a},t)\delta\eta(t_0-t)]n_j\,\mathrm{d}\sigma\ \mathrm{d}t\\
&-\int_G\Big|\rho\dot{x}_i(\mathbf{a},t)\delta x_i(t_0-t)-s(\mathbf{a},t)\delta\eta(t_0-t)\\
&-\tfrac{1}{2}\lambda_{ij}\eta_{,j}(\mathbf{a},t)\partial_i[\delta\eta(t_0-t)]\Big|_{t=0}^{t=t_0}\mathrm{d}^3a\\
&=-\delta A_G
\end{aligned}\right\} \tag{23b}
$$

In view of (2), the bracket (a) in (23) can be rewritten as

$$
J_m^{t_0}+\int_G\rho\dot{x}_i(\mathbf{a},t)x_{i,l}(\mathbf{a},t_0-t)\delta a_l\,\mathrm{d}^3a\Big|_{t=0}^{t=t_0}+\int_G s(\mathbf{a},0)\eta_{,l}(\mathbf{a},t_0)\,\delta a_l\,\mathrm{d}^3a. \tag{24}
$$

Let us point at the close analogy between the expression (24) and the impulse momentum type relation (1).

3. POTENTIAL ENERGY RELEASE RATE

From now on, our attention is restricted to quasistatic evolutions. Starting at time t we freeze the temperature increment field $\tau(\mathbf{a},\zeta)$ at its current value $\tau(\mathbf{a},t)$. We then compute the potential energy release rate associated with the *purely elastic* virtual change $\delta\bar{x}_k(\mathbf{a}, t)$ of the field variable $x_k(\mathbf{a}, t)$.

In our setting the internal energy functional is given by

$$\rho u(\mathbf{a}, t) = \tfrac{1}{2}c_{ijkh}(x_{k,h}(\mathbf{a}, t)-\delta_{kh})(x_{i,j}(\mathbf{a}, t)-\delta_{ij})+\tfrac{1}{2}\beta\dot{\eta}^2(\mathbf{a}, t). \tag{25}$$

The potential energy associated with the subdomain G is then given by

$$P_G(t) \overset{\text{def}}{=} \int_G \rho u(\mathbf{a}, t)\, \mathrm{d}^3a - \int_G \sigma_{ij}(\mathbf{a}, t)n_j(x_i(\mathbf{a}, t)-a_i)\, \mathrm{d}\sigma. \tag{26}$$

With the help of the divergence theorem and of the equilibrium equations, this last expression can be rewritten as

$$P_G(t) = \int_G \{-\tfrac{1}{2}c_{ijkh}(x_{k,h}(\mathbf{a}, t)-\delta_{kh})[x_{i,j}(\mathbf{a}, t)-\delta_{ij}]$$
$$+\beta_{ij}(x_{i,j}(\mathbf{a}, t)-\delta_{ij})\dot{\eta}(\mathbf{a}, t)+\tfrac{1}{2}\beta\dot{\eta}^2(\mathbf{a}, t)\}\, \mathrm{d}^3a. \tag{27}$$

Computing the variation $\delta P_G(t)$ of the expression (27), when the body B undergoes a *purely elastic* virtual change $\delta\bar{x}_k(\mathbf{a}, t)$ of the field variable $x_i(\mathbf{a}, t)$ is a trivial task if one bears in mind that $\dot{\eta}(\mathbf{a}, t)$ (i.e. the temperature increment field) remains constant during such an evolution.

We obtain:

$$\delta P_G(t) = -\int_G \sigma_{ij}(\mathbf{a}, t)(\delta\bar{x}_i)_{,j}(\mathbf{a}, t)\, \mathrm{d}^3a. \tag{28}$$

The divergence theorem, together with the *real* equilibrium equations at time t, yield

$$\delta P_G(t) = -\int_{\partial G} \sigma_{ij}(\mathbf{a}, t)\delta\bar{x}_i(\mathbf{a}, t)n_j\, \mathrm{d}\sigma. \tag{29}$$

Thus the total variation of $P_G(t)$ during the time interval $(0, t_0)$ is given by

$$\delta P'^{\,\mathscr{G}}_{G\,\text{total}} = \int_0^{t_0} \delta P_G(t)\, \mathrm{d}t = -\int_0^{t_0}\int_{\delta G} \sigma_{ij}(\mathbf{a}, t)\delta\bar{x}_i(\mathbf{a}, t)n_j\, \mathrm{d}\sigma\, \mathrm{d}t. \tag{30}$$

4. TRANSVERSALITY RELATION AND ENERGY RELEASE RATE

In this section we postulate the stationarity of the action under the one-parameter family of transformations described in Section 2, restricting our attention to quasistatic evolutions.

Remark

In the familiar case of a semi-infinite straight crack in an infinite homogeneous *thermoelastic* 2-dimensional body loaded at infinity, the action will be stationary under a transformation such that

$$\delta\mathbf{a} \text{ is parallel to } \boldsymbol{v}, \tag{31}$$

where $\boldsymbol{v}$ is a unit vector in the direction of the crack.

Setting δA_G to zero in eqn (23) yields a relation between the variations of the dependent and independent variables, i.e. between the δx_i's and $\delta\eta$ on the one hand and the δa_i's on the other hand. As such this relation is referred to as a transversality condition (see Edelen[5]).

With the help of the remark at the end of Section 2, this transversality condition can be written as

$$\begin{aligned} J_m'^{0} + \int_G s(\mathbf{a}, 0)\eta_{,l}(\mathbf{a}, t_0)\delta a_l \, d^3a \\ = -\int_{\partial G}\int_0^{t_0} [\sigma_{ij}(\mathbf{a}, t)\delta x_i(t_0-t) - q_j(\mathbf{a}, t)\delta\eta(t_0-t)]n_j \, d\sigma \, dt \\ - \int_G \left| s(\mathbf{a}, t)\delta\eta(t_0-t) + \tfrac{1}{2}\lambda_{ij}\eta_{,j}(\mathbf{a}, t)\,\partial_i(\delta\eta(t_0-t)) \right|_{t=0}^{t=t_0} d^3a. \end{aligned} \quad (32)$$

If the one-parameter family of transformations defined in Section 2 has the further property that

$$\delta\eta(t) = 0, \quad \text{for } 0 \leqslant t \leqslant t_0 \quad (33)$$

the transversality condition becomes

$$J_m'^{0} + \int_G s(\mathbf{a}, 0)\eta_{,l}(\mathbf{a}, t_0)\,\delta a_l \, d^3a = -\int_{\partial G}\int_0^{t_0} \sigma_{ij}(\mathbf{a}, t)\,\delta x_i(t_0-t)n_j \, d\sigma \, dt. \quad (34)$$

It is merely a matter of setting

$$\delta\bar{x}_i(t) = \delta x_i(t_0-t) \quad (35)$$

in (30) to obtain the right-hand side of eqn (34).

We have thus shown that, in a quasistatic setting, the quantity

$$J_m'^{0} + \int_G s(\mathbf{a}, 0)\eta_{,l}(\mathbf{a}, t_0)\,\delta a_l \, d^3a \quad (36)$$

equals the potential energy release rate associated with the purely elastic virtual change $\delta x_k(\mathbf{a}, t_0-t)$ of the field variable $x_k(\mathbf{a}, t)$.

If the initial entropy happens to be zero (which corresponds to the case where $J_m'^{0}$ is truly path-independent[1]) we recover in our setting the exact analogue of the elastic case.

As it had been noted in Ref. [1], the striking difference with the classical situations of fracture mechanics lies in the nonlocal character in time of the quantities defined.

5. CONCLUSIONS

In this paper we have established the close relation between path integrals of the material momenta $J_m'^{0}$ and a concept of potential energy release rate. In contrast with their elastic analogues, the quantities involved here are to be considered over a time interval, which underlines once again the importance of the dissipative character of a thermoelastic system.

It would of course remain to show that critical values of the potential energy release rate introduced in Section 3 constitute a sound criterion for the growth of a defect in an elastic medium where thermal dissipation is not negligible.

Acknowledgements—This work was supported in part by Air Force Grant AFOSR 80-0149 to Stanford University. One of the authors (G.F.) wishes to acknowledge D. Blanchard and J. P. Puel for most helpful discussions.

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